Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 1115–1123
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Balanced metrics on $\mathbb{C}^n$

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Received 5 October 2005; received in revised form 22 August 2006; accepted 4 September 2006
Available online 6 October 2006

Abstract

Let g be a Kähler metric on $\mathbb{C}^n$ and let $\mathcal{H}_\Phi$ be the complex Hilbert space consisting of global holomorphic functions f on $\mathbb{C}^n$ such that

$$\int_{\mathbb{C}^n} e^{-\Phi}|f|^2 d\mu(z) < \infty,$$

where $\Phi : \mathbb{C}^n \to \mathbb{R}$ is a Kähler potential for g and $d\mu(z)$ is the standard Lebesgue measure on $\mathbb{C}^n$. In this paper we prove that if (1) g is balanced with respect to the Euclidean metric, (2) $\Phi(z) = g_1(|z_1|^2) + \cdots + g_n(|z_n|^2)$ and (3) $z_1^{j_1} \cdots z_n^{j_n}$ belong to $\mathcal{H}_\Phi$, for all non-negative integers $j_1, \ldots j_n$, then, up to biholomorphic isometries, g equals the Euclidean metric. The proof is based on Calabi's diastasis function and on the characterization of the exponential function due to Miles and Williamson.

MSC: 53D05; 53C55; 58C25; 58F06

Keywords: Kähler metrics; Diastasis function; Exponential; Quantization

1. Introduction and statement of the main result

Let M be a n-dimensional complex manifold and let g be a Kähler metric on M polarized with respect to a holomorphic line bundle L over M, i.e. $c_1(L) = [\omega]$, where ω denotes the Kähler form associated with g. Further, let h be a Hermitian metric on L such that its Ricci curvature $\mathrm{Ric}(h) = \omega$, where $\mathrm{Ric}(h)$ is the two form on M whose local expression is given by

$$\mathrm{Ric}(h)(x) = -\frac{i}{2}\partial\bar{\partial}\log h(\sigma(x), \sigma(x)), \tag{1}$$

for a trivializing holomorphic section $\sigma : U \to L$. In the quantum mechanics terminology L is called the *quantum line bundle* and the pair (L, h) is called a *geometric quantization* of the Kähler manifold (M, g) (see e.g. [1]). Let g_0 be another Kähler metric on M. Consider the separable complex Hilbert space $\mathcal{H}_{h,g_0}$ consisting of global holomorphic

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2006.09.006

sections s of L which are bounded with respect to

$$\langle s, s\rangle_{h,g_0} = \int_M h(s(x), s(x))\frac{\omega_0^n}{n!}, \tag{2}$$

where ω_0 is the Kähler form associated with g_0.

Assume that for each point $x \in M$ there exists $s \in \mathcal{H}_{h,g_0}$ non-vanishing at x. Then one can consider the following holomorphic map into the N-dimensional ($N \leq \infty$) complex projective space:

$$\varphi_{(h,g_0)} : M \to \mathbb{C}P^N : x \mapsto [s_0(x), \ldots, s_N(x)], \tag{3}$$

where s_j, $j = 0, \ldots, N$, is a orthonormal basis for $(\mathcal{H}_{h,g_0}, \langle\cdot,\cdot\rangle_{h,g_0})$. In the case where $N = \infty$, $\mathbb{C}P^\infty$ define the quotient space of $l^2(\mathbb{C})$ (the space of sequences z_j such that $\sum_{j=1}^\infty |z_j|^2 < \infty$), where two sequences z_j and w_j are equivalent iff there exists $\lambda \in \mathbb{C}^*$ such that $\lambda z_j = w_j, \forall j$.

We say that the metric g is (h, g_0)-balanced (or simply g_0-balanced when the Hermitian metric is clear from the context) if $\varphi^*_{(h,g_0)}(g_{FS}) = g$, or equivalently

$$\varphi^*_{(h,g_0)}(\omega_{FS}) = \omega, \tag{4}$$

where g_{FS} is the Fubini–Study metric on $\mathbb{C}P^N$ and ω_{FS} its associated Kähler form. (Note that this definition is independent of the choice of the orthonormal basis.) Therefore, if g is a g_0-balanced metric, then g is projectively induced via the map (3). In the case where a metric g is g-balanced, one simply calls g a *balanced* metric. (For the balanced metrics the map $\varphi_{(h,g)}$ was introduced by Rawnsley [19] in the context of quantization of Kähler manifolds and it is often referred to as the *coherent states map*.)

The balanced and g_0-balanced metrics are important for the theories of quantization of Kähler manifolds and for the stability of complex vector bundles (see e.g. [1,2] and the reference therein). They are also deeply related to Kähler–Einstein metrics [20–22] and to the existence and uniqueness of extremal and constant scalar curvature metrics [4,5,16,17].

In the compact case the existence and the uniqueness of balanced (resp. g_0-balanced) metrics have been studied in [2,4,5] (resp. [23,24]).

The study of balanced and g_0-balanced metrics in the noncompact case is a very fruitful area of research (see [2,6, 10,11,13–15,19]). Nevertheless many questions on the uniqueness of balanced metrics are still open. For example, it is not known whether there exists a complete balanced metric on $\mathbb{C}^n$ different from the Euclidean one.

In this paper we study the g_0-balanced metrics g on $\mathbb{C}^n$, when $g_0 = g_{\mathrm{eucl}}$, the standard Euclidean metric on $\mathbb{C}^n$. Observe that any holomorphic line bundle on $\mathbb{C}^n$ is holomorphically trivial. Therefore we can assume, without loss of generality, that $L = \mathbb{C}^n \times \mathbb{C}$. Further, there exists a real valued function Φ on $\mathbb{C}^n$ (a Kähler potential for g) such that $\omega = \frac{\mathrm{i}}{2}\partial\bar\partial\Phi$. Finally, observe that the function $\mathrm{e}^{-\Phi}$ defines a Hermitian metric h on L on setting

$$h(z, t) = \mathrm{e}^{-\Phi(z)}|t|^2, \quad z \in \mathbb{C}^n,\ t \in \mathbb{C}.$$

It follows by (1) that the pair $(L, h = \mathrm{e}^{-\Phi})$ is indeed a geometric quantization of the Kähler manifold $(\mathbb{C}^n, g)$. In the case where g_0 equals the Euclidean metric g_{eucl}, the Hilbert space $\mathcal{H}_{h,g_0}$ consists of all holomorphic functions $f : \mathbb{C}^n \to \mathbb{C}$ (i.e. holomorphic sections of L) such that

$$\int_{\mathbb{C}^n} \mathrm{e}^{-\Phi}|f|^2 \mathrm{d}\mu(z) < \infty,$$

where $\mathrm{d}\mu(z) = \frac{\mathrm{i}^n}{2^n}\mathrm{d}z_1 \wedge \mathrm{d}\bar z_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar z_n$ is the standard Lebesgue measure on $\mathbb{C}^n$. In the sequel, we denote this Hilbert space by $\mathcal{H}_\Phi$ and the map (3) by φ_Φ.

Example 1.1. Let $\Phi(z) = |z|^2 = |z_1|^2 + \cdots + |z_n|^2$ and $h = \mathrm{e}^{-|z|^2}$. Then the Hilbert space $\mathcal{H}_\Phi(\Phi(z) = |z|^2)$ consists of all the holomorphic functions on $\mathbb{C}^n$ such that $\int_{\mathbb{C}^n} \mathrm{e}^{-|z|^2}|f|^2\mathrm{d}\mu(z) < \infty$. One can easily verify that $\frac{z^{m_j}}{\sqrt{\pi^n m_j!}}$, $j = 0, \ldots$ is an orthonormal basis of $\mathcal{H}_\Phi$, where $m_j! = m_{j_1}! \cdots m_{j_n}!$ (see Remark 2.3 below for the notation). Therefore the map (3) in this case is given by

$$\varphi_\Phi : \mathbb{C}^n \to \mathbb{C}P^\infty : z = (z_1, \dots, z_n) \mapsto \left[\dots, \frac{z^{m_j}}{\sqrt{\pi^n m_j!}}, \dots\right].$$

Thus,

$$\varphi_\Phi^*(g_{\mathrm{FS}}) = \frac{\mathrm{i}}{2}\partial\bar\partial \log\left(\frac{1}{\pi^n}\sum_{j=0}^{+\infty}\frac{|z|^{2m_j}}{m_j!}\right) = \frac{\mathrm{i}}{2}\partial\bar\partial \log \mathrm{e}^{|z|^2} = \omega_{\mathrm{eucl}}$$

and so g is a g_{eucl}-balanced (even balanced) metric.

The main result of this paper is the following:

Theorem 1.2. *Let g be a g_0-balanced metric on $\mathbb{C}^n$ where $g_0 = g_{\mathrm{eucl}}$ and let $(L, h = \mathrm{e}^{-\Phi})$ be a geometric quantization of $(\mathbb{C}^n, g)$ as above. Assume that:*

(i) *the metric g admits a (globally) defined Kähler potential Φ of the form*

$$\Phi(z) = g_1(|z_1|^2) + \cdots + g_n(|z_n|^2), \tag{5}$$

where $g_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, n$, and $z = (z_1, \dots, z_n)$ are the Euclidean coordinates on $\mathbb{C}^n$;

(ii) *$z_1^{j_1} \cdots z_n^{j_n}$ belong to $\mathcal{H}_\Phi$, for all non-negative integers $j_1, \dots j_n$.*

Then, up to biholomorphic isometries, $g = g_{\mathrm{eucl}}$.

Remark 1.3. Notice that the metric g_0 only enters in the proof of Theorem 1.2 through its volume. Indeed, from its proof (see Section 3) it follows that the hypothesis on g_0 of being the Euclidean metric can be replaced by the weaker assumption that g_0 is a Kähler metric on $\mathbb{C}^n$ having the same volume form of the Euclidean metric, i.e. satisfying

$$\frac{\omega_0^n(z)}{n!} = \mathrm{d}\mu(z). \tag{6}$$

In this regard, it is worth pointing out that there exist examples of noncomplete Kähler metrics g_0 on $\mathbb{C}^n$ ($n \geq 2$) satisfying Eq. (6) and it is conjecturally true that the only complete metric on $\mathbb{C}^n$ satisfying this equation is the Euclidean one (see [25]).

Despite the very strong assumption (5) on the Kähler potential, the proof of our theorem is far from being trivial. Indeed, it is based on: (1) Calabi's diastasis function (see Section 2) which comes into the game due to the fact that g_0-balanced metrics are projectively induced, namely they satisfy Eq. (4); (2) a characterization of the exponential function (see Section 3) which has been an open conjecture for almost twenty years and was finally proved in 1986 by Miles and Williamson [18].

The paper is organized as follows. In Section 2 we describe Calabi's work on the diastasis and on the holomorphic and isometric immersions into complex projective spaces. In particular we get Lemma 2.8 which is one of the main ingredients in the proof of Theorem 1.2. In Section 3 we recall the characterization of the exponential function due to Miles and Williamson, we obtain Lemma 3.1, Corollary 3.2 and we prove our main result Theorem 1.2.

2. Calabi's diastasis function

In his seminal paper Calabi [3] gave a complete answer to the problem of the existence and uniqueness of holomorphic and isometric immersions of a Kähler manifold (M, g) into a finite or infinite dimensional complex projective space $(\mathbb{C}P^N, g_{\mathrm{FS}})$, $N \leq \infty$, where g_{FS} denotes the Fubini–Study metric on $\mathbb{C}P^N$ (see Example 2.2 below).

Calabi's first observation was that if a holomorphic and isometric immersion (M, g) into a complex projective space exists then the metric g is forced to be real analytic being the pull-back via a holomorphic map of the real analytic metric g_{FS}. Then in a neighborhood of every point $p \in M$, one can introduce a very special Kähler potential D_p^g for the metric g, which Calabi named *diastasis*. Recall that a Kähler potential for a smooth metric g is a smooth function Φ defined in a neighborhood of a point p such that $\omega = \frac{\mathrm{i}}{2}\bar\partial\partial\Phi$, where ω is the Kähler form associated with

g. A Kähler potential is not unique: it is defined up to an addition of the real part of a holomorphic function. If the metric g is real analytic, then Φ can taken real analytic. In this case, by duplicating the variables z and $\bar{z}$ a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$ denotes the manifold conjugate to M). The *diastasis function* is the Kähler potential D_p^g around p defined by

$$D_p^g(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Example 2.1. Let g_{eucl} be the Euclidean metric on $\mathbb{C}^N$, $N \leq \infty$, namely the metric whose associated Kähler form is given by $\omega_{\mathrm{eucl}} = \frac{\mathrm{i}}{2} \sum_{j=1}^N \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$. Here $\mathbb{C}^\infty$ is the complex Hilbert space $l^2(\mathbb{C})$ consisting of sequences z_j, $j = 1 \ldots, z_j \in \mathbb{C}$ such that $\sum_{j=1}^{+\infty} |z_j|^2 < +\infty$. The diastasis function $D_p^{g_{\mathrm{eucl}}} : \mathbb{C}^N \to \mathbb{R}$ around $p \in \mathbb{C}^N$ is given by the square of the distance between p and q, i.e. $D_p^{g_{\mathrm{eucl}}}(q) = \sum_{j=1}^N |p_j - q_j|^2$.

Example 2.2. Let $(Z_0, Z_1, \ldots, Z_N)$ be the homogeneous coordinates in $\mathbb{C}P^N$, $N \leq \infty$, and let $p = [1, 0, \ldots, 0]$. In the affine chart $U_0 = \{Z_0 \neq 0\}$ endowed with coordinates $(z_1, \ldots, z_n)$, $z_j = \frac{Z_j}{Z_0}$ the diastasis around p reads as

$$D_p^{g_{\mathrm{FS}}}(z) = \log\left(1 + \sum_{j=1}^n |z_j|^2\right). \tag{7}$$

A very useful characterization of the diastasis can be obtained as follows. Let (z) be a system of complex coordinates in a neighbourhood of p where D_p^g is defined and consider its power series development:

$$D_p^g(z) = \sum_{j,k \geq 0} a_{jk}(g) z^{m_j} \bar{z}^{m_k}. \tag{8}$$

Remark 2.3. Here, and throughout this paper, we are using the following convention: we arrange every n-tuple of non-negative integers as the sequence $m_j = (m_{1,j}, m_{2,j}, \ldots, m_{n,j})_{j=0,1,\ldots}$ such that $m_0 = (0, \ldots, 0)$, $|m_j| \leq |m_{j+1}|$, with $|m_j| = \sum_{\alpha=1}^n m_{\alpha,j}$ and $z^{m_j} = \prod_{\alpha=1}^n (z_\alpha)^{m_{\alpha,j}}$. Further, we order all the m_j's with the same $|m_j|$ using the lexicographic order in the variables $(z_1, \ldots, z_n)$. Our notation (due to Calabi [3]) is more suitable for describing the matrices appearing in (8) and (9) below than the usual multi-index notation.

Characterization of the diastasis: *Among all the potentials the diastasis is characterized by the fact that in every coordinates system (z) centered in p the coefficients $a_{jk}(g)$ of the expansion* (8) *satisfy $a_{j0}(g) = a_{0j}(g) = 0$ for every non-negative integer j.*

The diastasis function is the key tool for studying holomorphic and isometric immersions (i.e. Kähler immersions) into finite or infinite dimensional complex projective spaces due to its hereditary property:

Theorem 2.4 (*Calabi [3]*). *Let (M, g) be a Kähler manifold which admits a Kähler immersion $\varphi : (M, g) \to (\mathbb{C}P^N, g_{\mathrm{FS}})$, $N \leq \infty$. Then*

(1) *the metric g is real analytic;*
(2) *$D_{\varphi(p)}^{g_{\mathrm{FS}}} \circ \varphi = D_p^g$, where both sides are defined.*

One of the main ingredients in the proof of our Theorem 1.2 is Theorem 2.7 and its corollary (Lemma 2.8) below, which goes deeply to the heart of the problem we are dealing with. In order to state it we need some definitions due to Calabi.

Definition 2.5. A holomorphic and isometric immersion φ of (M, g) into $(\mathbb{C}P^\infty, g_{\mathrm{FS}})$ is said to be full if $\varphi(M)$ is not contained in a proper complex projective subspace of $\mathbb{C}P^\infty$.

Definition 2.6. Consider the function $\mathrm{e}^{D_p^g} - 1$ and its power series development:

$$\mathrm{e}^{D_p^g} - 1 = \sum_{j,k \geq 0} b_{jk}(g) z^{m_j} \bar{z}^{m_k}. \tag{9}$$

The metric g is said to be ∞-*resolvable* at p if the $\infty \times \infty$ matrix $b_{jk}(g)$ is positive semidefinite and of infinity rank.

Theorem 2.7 (*Calabi*). *Let M be a complex manifold endowed with a real analytic Kähler metric g.*

(i) *A neighbourhood of a point p admits a full holomorphic and isometric immersion into $(\mathbb{C}P^\infty, g_{FS})$ if and only if g is ∞-resolvable at p.*
(ii) *Two full holomorphic and isometric immersions $\varphi : (M, g) \to (\mathbb{C}P^\infty, g_{FS})$ and $\psi : (M, g) \to (\mathbb{C}P^\infty, g_{FS})$ are congruent, i.e. there exists an unitary transformation U of $(\mathbb{C}P^\infty, g_{FS})$ such that $U \circ \varphi = \psi$.*

The previous theorem can be applied when one has the explicit expression of the diastasis function and when the matrix $b_{jk}(g)$ in Definition 2.6 is not too complicated. In some very special cases like for the complex space forms [3] or for Hartogs domains [12] or more generally when the diastasis is rotation invariant, namely it depends only on $|z_1|^2, \ldots, |z_n|^2$, the matrix $b_{jk}(g)$ (given by (9)) is diagonal, i.e.

$$b_{jk}(g) = b_j \delta_{jk}, \quad b_j \in \mathbb{R}. \tag{10}$$

In this case (i) of Theorem 2.7 reads:

Lemma 2.8. *Let g be a real analytic Kähler metric on a complex manifold M and assume that there exist complex coordinates in a neighborhood of a point $p \in M$ such that its diastasis function $D_p^g : U \to \mathbb{R}$ is rotation invariant. Then there exists a holomorphic and isometric immersion φ of (U, g) into $(\mathbb{C}P^\infty, g_{FS})$ if and only if all the b_j's given by (10) are greater or equal than 0 and an infinite number of them are positive.*

3. A characterization of the exponential and the proof of Theorem 1.2

Let

$$f(t) = \sum_j b_j t^{m_j}, \quad t^{m_j} = t_1^{m_{j1}} \cdots t_n^{m_{jn}} \tag{11}$$

be an entire function (see Remark 2.3 for the multi-index notation). Assume that

$$b_0 = 1, \qquad b_j > 0, \quad \forall j \tag{12}$$

and

$$\int_{\mathbb{R}^n_+} \frac{b_j t^{m_j}}{f(t)} \, \mathrm{d}t = 1, \quad \forall j, \tag{13}$$

where $\mathbb{R}^n_+ = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : t_i \geq 0, \forall i\}$ and $\mathrm{d}t = \mathrm{d}t_1 \cdots \mathrm{d}t_n$. When $n = 1$ it is straightforward to verify that conditions (11) and (13) are satisfied by the exponential function e^t. Rényi and Vincze conjectured that e^t is the only such function ([7], Problem 2.32). Some partial results were obtained by various authors (see [8,9]). Finally, Miles and Williamson [18] gave a positive answer to the conjecture of Rényi and Vincze. For $n \geq 2$ the classification of all the functions satisfying the above conditions is a challenging problem. In the case where $f(t)$ is a product, i.e.

$$f(t_1, \ldots, t_n) = g_1(t_1) \cdot g_2(t_2) \cdot \cdots \cdot g_n(t_n) \tag{14}$$

where $g_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, n$, by using the above theorem of Miles and Williamson we have the following:

Lemma 3.1. *Assume $f(t)$ is a real valued function on $\mathbb{R}^n$ satisfying (11)–(14). Then there exist positive constants $c_1, c_2, \ldots, c_n$ with $c_1 \cdot c_2 \cdot \cdots \cdot c_n = 1$ such that*

$$f(t_1, \ldots, t_n) = \mathrm{e}^{c_1 t_1 + c_2 t_2 + \cdots + c_n t_n}. \tag{15}$$

Proof. In order to prove the lemma we use the standard multi-index notation. So, let $I = (i_1, \ldots, i_n)$ be a multi-index of non-negative integers. Conditions (11)–(13) read

$$f(t) = \sum_I B_I t^I, \quad B_I = B_{i_1, \ldots, i_n}, \tag{16}$$

$$B_0 = B_{0,\ldots,0} = 1, \quad B_I > 0, \ \forall I \tag{17}$$

and

$$\int_{\mathbb{R}_+^n} \frac{B_I t^I}{f(t)}\, \mathrm{d}t = 1, \quad \forall I. \tag{18}$$

Set

$$c_i = \left(\int_0^\infty \frac{g_i(0)\, \mathrm{d}t_i}{g_i(t_i)}\right)^{-1} \quad \text{and} \quad h_i(s_i) = \frac{g_i(\frac{s_i}{c_i})}{g_i(0)} \quad \text{for } i = 0, 1, \ldots, n.$$

Note that we have $c_1 \cdot c_2 \cdots c_n = 1$ by

$$g_1(0) \cdot g_2(0) \cdots g_n(0) = f(0) = 1$$

and

$$\int_0^\infty \frac{\mathrm{d}t_1}{g_1(t_1)} \cdot \int_0^\infty \frac{\mathrm{d}t_2}{g_2(t_2)} \cdots \int_0^\infty \frac{\mathrm{d}t_n}{g_n(t_n)} = 1.$$

The proof of the lemma will be obtained if we show that

$$h_i(s_i) = \mathrm{e}^{s_i}. \tag{19}$$

Indeed this is equivalent to $g_i(t_i) = g_i(0)\mathrm{e}^{c_i t_i}$, which, combined with (14), implies (15). In order to prove (19) observe that by (16) the power series of h_i is

$$\begin{aligned} h_i(s_i) &= \frac{g_i(\frac{s_i}{c_i})}{g_i(0)} = \frac{1}{g_i(0) \prod_{j \neq i} g_j(0)} f\left(0, \ldots, \frac{s_i}{c_i}, \ldots, 0\right) \\ &= f\left(0, \ldots, \frac{s_i}{c_i}, \ldots, 0\right) = \sum_{k=0}^\infty B_k \frac{s_i^k}{c_i^k}, \end{aligned}$$

where $B_k = B_{0,\ldots,0,k,0,\ldots,0}$. Since by (17) $B_0 = 1$ and $\frac{B_k}{c_i^k} > 0\ \forall k = 1, 2, \ldots$, in order to apply the above mentioned theorem of Miles and Williamson one needs to verify the following equalities:

$$\int_0^\infty \frac{s_i^k}{h_i(s_i)}\, \mathrm{d}s_i = \frac{c_i^k}{B_k}, \quad \forall k = 1, 2, \ldots. \tag{20}$$

Indeed, by the change of variable $t_i = s_i/c_i$ and by assumption (18) one gets

$$\begin{aligned} \int_0^\infty \frac{s_i^k}{h_i(s_i)}\, \mathrm{d}s_i &= \int_0^\infty \frac{c_i^k t_i^k c_i\, \mathrm{d}t_i}{h_i(t_i c_i)} = c_i^k c_i g_i(0) \int_0^\infty \frac{t_i^k}{g_i(t_i)}\, \mathrm{d}t_i \\ &= c_i^k \int_0^\infty \frac{\mathrm{d}t_1}{g_1(t_1)} \cdot \int_0^\infty \frac{\mathrm{d}t_2}{g_2(t_2)} \cdots \int_0^\infty \frac{t_i^k\, \mathrm{d}t_i}{g_i(t_i)} \cdots \int_0^\infty \frac{\mathrm{d}t_n}{g_n(t_n)} \\ &= c_i^k \int_{\mathbb{R}_+^n} \frac{t_i^k\, \mathrm{d}t}{f(t)} = \frac{c_i^k}{B_k}, \quad \forall k = 1, 2, \ldots. \end{aligned}$$

We are interested in a slightly general assumption on $f(t)$, namely instead of (13) we assume that

$$\int_{\mathbb{R}_+^n} \frac{\lambda b_j t^{m_j}}{f(t)}\, \mathrm{d}t = 1, \quad \forall j \tag{21}$$

for some fixed $\lambda > 0$. As a consequence of the previous lemma one gets the following corollary which, together with Lemma 2.8 above, is one of the main ingredients in the proof of Theorem 1.2. □

Corollary 3.2. *If $f(t)$ satisfies (11), (12), (14) and (21) for some $\lambda > 0$ then there exist positive constants $c_1, c_2, \ldots, c_n$ with $c_1 \cdot c_2 \cdots c_n = \lambda$ such that*

$$f(t_1, \ldots, t_n) = \mathrm{e}^{c_1 t_1 + c_2 t_2 + \cdots + c_n t_n}.$$

Proof. Consider the function $h(s) = f(s/\sqrt[n]{\lambda})$. We claim that $h(s)$ satisfies the four hypotheses of the previous lemma. Indeed $h(s)$ is an entire function with

$$h(s) = \sum_j \frac{b_j}{\lambda^{|m_j|/n}} s^{m_j}, \qquad b_0 = 1, \quad b_j > 0,$$

and so (11) and (12) hold true for $h(s)$. By the change of variables $t = s/\sqrt[n]{\lambda}$ and by (21) we get

$$\int_{\mathbb{R}^n_+} \frac{b_j s^{m_j}}{\lambda^{|m_j|/n} h(s)} \mathrm{d}s = \int_{\mathbb{R}^n_+} \frac{\lambda b_j t^{m_j}}{f(t)} \mathrm{d}t = 1, \quad \forall j.$$

Thus (13) is valid for $h(s)$. Obviously, by (14), $h(s)$ splits as

$$h(s) = g_1(s_1/\sqrt[n]{\lambda}) \cdots g_n(s_n/\sqrt[n]{\lambda}).$$

Therefore, by Lemma 3.1 we have

$$h(s) = \mathrm{e}^{d_1 s_1 + d_2 s_2 + \cdots + d_n s_n}$$

with $d_i > 0$ for each i, and $d_1 \cdot d_2 \cdots d_n = 1$. Setting $c_i = \sqrt[n]{\lambda} d_i$ we conclude that

$$f(t) = h(\sqrt[n]{\lambda} t) = \mathrm{e}^{c_1 t_1 + c_2 t_2 + \cdots + c_n t_n},$$

with $c_1 \cdot c_2 \cdots c_n = \lambda$. □

Proof of Theorem 1.2. The Kähler metric g is g_0-balanced ($g_0 = g_{\mathrm{eucl}}$) if there exists a sequence of holomorphic functions f_j on $\mathbb{C}^n$, with $\sum_j |f_j|^2 \neq 0$, which are an orthonormal basis for $\mathcal{H}_\Phi$, i.e.

$$\int_{\mathbb{C}^n} \mathrm{e}^{-\Phi} f_j \overline{f_k} \mathrm{d}\mu(z) = \delta_{jk} \tag{22}$$

and such that

$$\varphi_\Phi^*(\omega_{\mathrm{FS}}) = \frac{\mathrm{i}}{2} \partial\bar{\partial} \log\left(\sum_{j=0}^{\infty} |f_j|^2\right) = \omega \tag{23}$$

where the map $\varphi_\Phi : \mathbb{C}^n \to \mathbb{C}P^\infty$ is given by taking the equivalence class $[\ldots, f_j, \ldots] \in \mathbb{C}P^\infty$ of $(\ldots, f_j, \ldots) \in l^2(\mathbb{C}) \setminus \{0\}$ (see (3)). Observe also that φ_Φ is full since the f_j's are linearly independent. By hypothesis (ii) the constant functions belong to $\mathcal{H}_\Phi$ and so we can assume that $f_0 = 1$. Further, without loss of generality, up to a unitary transformation of $\mathbb{C}P^\infty$, we can assume that

$$\varphi_\Phi(0, \ldots, 0) = [1, 0, \ldots, 0]. \tag{24}$$

From (23) and Theorem 2.4 the metric g is real analytic and the diastasis function of the metric g around the origin is given by

$$D_0^g(z) = \log\left(1 + \sum_{j=1}^{\infty} |f_j(z)|^2\right). \tag{25}$$

Assumption (5) implies that its Taylor expansion at the origin is of the form

$$\Phi(z) = \sum_{j=0}^{\infty} a_j |z|^{2m_j}, \quad z = (z_1, \ldots, z_n),\ a_j \in \mathbb{R}.$$

It follows by the characterization of the diastasis (after Remark 2.3 in Section 2) that the function $D : \mathbb{C}^n \to \mathbb{R}$ given by $D(z) = \Phi(z) - a_0$ is indeed the diastasis function for the metric g around the origin, i.e. $D = D_0^g$. Hence by formula (25) one gets

$$\mathrm{e}^{-\Phi(z)} = \frac{\mathrm{e}^{-a_0}}{1 + \sum_{j=1}^{\infty} |f_j(z)|^2}. \tag{26}$$

Furthermore, if

$$e^{D_0^g(z)} - 1 = \sum_{j=1}^{\infty} b_j |z|^{2m_j}, \quad b_j \in \mathbb{R} \tag{27}$$

is the Taylor expansion of the real analytic function $e^{D_0^g(z)} - 1$ at the origin, it follows by Lemma 2.8 that $b_j \geq 0$ for all j and there exists an infinite sequence of b_j's different from zero. Let $J \subset \mathbb{N}$ be the set of all j's ($j \geq 1$) such that b_j is strictly greater than zero. Set $\beta_l = \sqrt{b_l}$ for $l \in J$ and consider the full holomorphic map of $\mathbb{C}^n$ into $\mathbb{C}P^\infty$ given by

$$\psi : \mathbb{C}^n \to \mathbb{C}P^\infty : z = (z_1, \ldots, z_n) \mapsto [1, \ldots, \beta_l z^{m_l}, \ldots]. \tag{28}$$

This map is a isometric. Indeed,

$$\psi^*(\omega_{FS}) = \frac{i}{2} \partial \bar{\partial} \log \left(1 + \sum_{l \in J} b_l |z|^{2m_l} \right) = \frac{i}{2} \partial \bar{\partial} \log e^{D_0^g} = \omega.$$

From (ii) in Theorem 2.7 applied to the maps φ_Φ and ψ we get that there exists a non-vanishing holomorphic function k on $\mathbb{C}^n$ such that the vectors $(1, \ldots f_j(z), \ldots)$ and $k(z)(1, \ldots \beta_l z^{m_l}, \ldots)$ are related by a unitary transformation of $l^2(\mathbb{C})$. By (26) and (27) it follows that $|k|^2 = 1$ and by the open mapping theorem k is a constant. This implies that $(1, \ldots \beta_l z^{m_l}, \ldots)$ is an orthonormal basis for the Hilbert space $\mathcal{H}_\Phi$. In particular, since by hypothesis (ii) z^{m_j} belongs to $\mathcal{H}_\Phi$ for all j,

$$\int_{\mathbb{C}^n} e^{-a_0} \frac{b_j |z|^{2m_j}}{f(z)} d\mu(z) = 1, \quad \forall j, \tag{29}$$

where $f(z) = e^{D_0^g(z)} = 1 + \sum_{j=1}^{\infty} b_j |z|^{2m_j}$.

By taking polar coordinates $z_\alpha = \rho_\alpha e^{i\theta_\alpha}, \alpha = 1, \ldots, n, \rho_\alpha \in [0, +\infty), \theta_\alpha \in [0, 2\pi)$ and by the change of variables $t_\alpha = \rho_\alpha^2$ one gets

$$\int_{\mathbb{R}_+^n} \pi^n e^{-a_0} \frac{b_j t^{m_j}}{f(t)} dt = 1, \quad \forall j$$

where

$$f(t) = 1 + \sum_{j=1}^{\infty} b_j t^{m_j}.$$

By assumption (5) and Corollary 3.2 we get $\Phi(z) = a_0 + c_1|z_1|^2 + \cdots + c_n|z_n|^2$ (with $c_1 \cdots c_n = \pi^n e^{-a_0}$) and therefore the metric g is biholomorphically isometric to the Euclidean metric g_{eucl} of $\mathbb{C}^n$ via the (linear) map $F : (\mathbb{C}^n, g) \to (\mathbb{C}^n, g_{eucl}), z = (z_1, \ldots, z_n) \mapsto (\frac{z_1}{\sqrt{c_1}}, \ldots, \frac{z_n}{\sqrt{c_n}})$. $\square$

Remark 3.3. We conjecture that our result holds true also for the more general class of rotation invariant metrics g on $\mathbb{C}^n$, namely those metrics which admit a Kähler potential which depends only on $|z_1|^2, \ldots, |z_n|^2$ not necessarily satisfying Eq. (5). This amounts to proving a result similar to that of Corollary 3.2 without assumption (5).

Acknowledgment

The authors would like to thank the anonymous referee for useful remarks which helped them to improve the exposition.

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